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A THEOREM ON MINIMAL SURFACES

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1. Introduction

It is well known that $x\colon M^k \to \boldsymbol{R}^{k+p}$ is a minimal immersion into the Euclidean space $\boldsymbol{R}^{k+p}$ if and only if

$$\Delta\langle a, x\rangle = 0 \qquad \text{for all vectors } a \in \boldsymbol{R}^{k+p} ,$$

where Δ denotes the Laplacian on M^k. This result follows immediately from the relation

$$(1) \qquad \Delta\langle a, x\rangle = -\langle a, H\rangle ,$$

where H is the mean curvature vector of the immersion. Osserman [3, Theorem 2] proved the following theorem:

If $x\colon M^2 \to \boldsymbol{R}^3$ is such that for some $a \in \boldsymbol{R}^3$, $\Delta\langle a, x\rangle = 0$, then M^2 is a minimal surface, or else a locally cylindrical surface with its generators parallel to a.

The purpose of this paper is to generalize Osserman's theorem to the case $x\colon M^2 \to \boldsymbol{R}^n$. We will assume as a natural generalization of Osserman's hypothesis the existence of an $n-2$ dimensional subspace $A \subset \boldsymbol{R}^n$ so that $\Delta\langle a, x\rangle = 0$ for all $a \in A$. The main theorem is the following:

Theorem. *Let M^2 be a connected manifold, and A an $n-2$ dimensional subspace of Euclidean space $\boldsymbol{R}^n$. If $x\colon M^2 \to \boldsymbol{R}^n$ is an immersion such that*

$$(2) \qquad \Delta\langle a, x\rangle = 0 \qquad \text{for all } a \in A ,$$

then there are only two possibilities:

1. *M^2 is minimally immersed in R^n, or*
2. *there is a set $\{m_i\}$ of isolated points in M so that each point of the complement M_0 of this set has a neighborhood U which is mapped onto a regular curve γ in B, the orthogonal complement of A, by the orthogonal projection π with kernel A in such a way that $x|U$ is a minimal immersion in the cylinder $\gamma \times A$.*

Conversely, if the immersion $x\colon M^2 \to \boldsymbol{R}^n$ is as described under 1 or 2, it satisfies the hypothesis of the theorem.

The type of immersion described under point 2 will for convenience be referred to as type 2. Similarly for type 1.

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It would be natural to ask whether for type 2 the immersions $x|U$ of U into a piece of cylinder $\gamma \times A$ can be pieced together to give a factorization of the global immersion

$$x: M^2 \to R^n$$

as the product of two immersions, the first a minimal immersion of M^2 into a cylinder, and the second an immersion of the cylinder into R^n. This would give a much nicer characterization of the immersions of type 2. In particular, the set $\{m_i\}$ of isolated points would be empty.

However, an example at the end of this paper gives an immersion x of a complete manifold M^2 into R^4 satisfying the hypothesis (2) of the theorem and belonging to type 2, for which such a description is not possible. In fact, there are two points $m_i \in M^2$, $i = 1, 2$, in the example so that no neighborhood U of m_i is such that $x|U$ can be factored in the way suggested. Thus there does not seem to be a nicer way to characterize the immersions of type 2 than that already given in the statement of the theorem.

It will be clear in the proofs below that in the case of codimension one the points m_i do not occur, so that in that case we have precisely Osserman's theorem.

The theorem has the following easy corollary:

Corollary. *Let M^2 be a connected manifold, and A an $n - 1$ dimensional subspace of Euclidean space R^n. If x: $M^2 \to R^n$ is an immersion such that*

$$\Delta\langle a, x\rangle = 0 \qquad \text{for all } a \in A ,$$

then x is a minimal immersion.

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2. The function μ

We begin with some general remarks about H and Δ. For their definitions we follow [4]. Suppose we are given an immersion x: $M^k \to V^{k+p}$, where M^k and V^{k+p} are Riemannian manifolds. If $\{e_i\}$, $i = 1, \cdots, k$, is a tangent frame to the immersion, and $\{e_\alpha\}$, $\alpha = k + 1, \cdots, k + p$, a normal frame, the mean curvature vector is

$$H = \sum_\alpha \sum_i \langle \nabla_i e_\alpha, e_i\rangle e_\alpha , \tag{3}$$

where ∇ is the Riemannian connection in V^{k+p}, and ∇_i denotes the covariant derivative with respect to e_i. We get a simpler formula if we choose the vector fields e_i with the added condition that at one fixed point m we have

$$\bar{\nabla}_i e_j(m) = 0 , \tag{4}$$

where $\bar{\nabla}$ is the connection induced on M^k by the immersion. This is always possible. Then H has the following formula at the point m only:

$$H(m) = -\sum_i \nabla_i e_i(m) . \tag{5}$$

Using a frame field with the property (4) the Laplace operator Δ also has a simple expression at the point m: If u is a function,

$$\Delta u(m) = \sum_i e_i e_i u(m) . \tag{6}$$

This formula is given in [4, Prop. 1.2.1]. It is now easy to prove (1), for

$$\begin{aligned} \Delta\langle a, x\rangle(m) &= \sum_i e_i e_i \langle a, x\rangle(m) = \sum_i e_i \langle a, e_i\rangle(m) \\ &= \langle a, \sum_i \nabla_i e_i\rangle(m) , \end{aligned}$$

while

$$-\langle a, H\rangle(m) = \langle a, \sum_i \nabla_i e_i\rangle(m) .$$

In the case $k = 2$ it is well known that isothermal coordinates may be found in the neighborhood of any point of M^k. In such coordinates the mean curvature vector has a particularly nice expression which holds throughout the coordinate neighborhood. We give it in the following lemma, for it will be used later.

Lemma 1. *Let (u^1, u^2) be isothermal coordinates on a neighborhood U of a two-dimensional manifold, and $x\colon U \to V^n$ an immersion into a Riemannian manifold V^n. Then the mean curvature vector of the immersion is given by*

$$H = (-1/\langle v_1, v_1\rangle) \sum_{i=1}^{2} \nabla_i v_i , \tag{7}$$

where $v_i = \partial/\partial u^i$, ∇ is the connection in V^n, and ∇_i denotes the covariant derivative with respect to v_i.

Proof. Let e_α be a normal frame for the immersed neighborhood. Since $\|v_1\| = \|v_2\|$, it is easy to see that by putting

$$e_i = (1/\|v_1\|)v_i , \qquad i = 1, 2 ,$$

in (3) we obtain the following formula:

$$\begin{aligned} -H &= (1/\langle v_1, v_1\rangle) \sum_\alpha \sum_i \langle \nabla_i v_i, e_\alpha\rangle e_\alpha \\ &= (1/\langle v_1, v_1\rangle)(\sum_i \nabla_i v_i)^N , \end{aligned}$$

where $(\sum_i \nabla_i v_i)^N$ denotes the component orthogonal to U. It is therefore enough to show that for isothermal coordinates the component of $\sum_i \nabla_i v_i$ tangent to U vanishes. But, if $\bar{\nabla}$ is the connection induced on U, then

$$\nabla_i v_i = \bar{\nabla}_i v_i + (\nabla_i v_i)^N ,$$

so that we must show that $\sum_i \bar{\nabla}_i v_i$ vanishes. This is done directly. The Riemannian metric on U is given by

$$g_{11} = g_{22} = c , \qquad g_{12} = g_{21} = 0 .$$

From this we easily see that

$$\bar{\nabla}_1 v_1 = (c_1/2c)v_1 - (c_2/2c)v_2 ,$$
$$\bar{\nabla}_2 v_2 = -(c_1/2c)v_1 + (c_2/2c)v_2 ,$$

where $c_i = \partial c/\partial u^i$, $i = 1, 2$. Thus $\sum_i \bar{\nabla}_i v_i = 0$, which completes the proof.

Throughout the remainder of the paper we will assume the hypothesis (2) of the theorem. If T_m is the plane through the origin in $\boldsymbol{R}^n$ parallel to the tangent plane of the immersion at $m \in M$, we can define a function μ on M by

$$\mu(m) = \dim (T_m \cap A) .$$

Then $\mu = 0, 1$, or 2. It is easy to see that μ is upper semicontinuous in the sense that each $m \in M$ has a neighborhood U such that $\mu(m') \leq \mu(m)$ for all $m' \in U$. It will be shown later that the behavior of the function μ determines whether the immersion belongs to type 1 or type 2. The purpose of the three propositions in this section is to show that the behavior of μ cannot be characteristic of type 1 in one part of M and characteristic of type 2 in another part. Our main tool in the proofs is the complex structure on M^2 which is induced by the Riemannian structure via the existence of isothermal coordinates. A strong theorem of Rado on holomorphic functions enters crucially. Before we state the next lemma it should be remarked that it follows from (1) that the hypothesis (2) is equivalent to

$$(8) \qquad H(m) \perp A \qquad \text{for all } m \in M .$$

Here $H(m)$ is the mean curvature vector at m.

Lemma 2. *If the hypothesis* (2) *of the theorem holds, then* $\mu(m) = 0$ *implies* $H(m) = 0$.

Proof. If $\mu(m) = 0$, T_m and A are complementary subspaces of $\boldsymbol{R}^n$. But we have by (8), which is equivalent to (2), $H(m) \perp A$. It is always true that $H(m) \perp T_m$. Hence $H(m) = 0$, which completes the proof.

Proposition 1. *If* (8) *holds, and* $\mu(m) \geq 1$ *on some open set, then* $\mu(m) \geq 1$ *throughout* M.

Proof. Let C be the interior of the subset of M given by $\mu(m) \geq 1$, and let $C_0 = M - \bar{C}$. Because M is connected, if ∂C denotes the boudary of C, then clearly

$$\partial C = \phi \Rightarrow \bar{C} = C \Rightarrow C = M \quad \text{or} \quad C = \phi . \tag{9}$$

Let $m \in \partial C$, and let U be any neighborhood of m. We claim that the intersections of U with C and with $M - C$ both have nonempty interiors. Since C is open, one half of this claim is obvious. For the other half, suppose to the contrary that $U \cap (M - C) = U - C$ has empty interior. Then $U \cap C$ is dense in U. By the upper semi-continuity of μ this implies that $\mu \geq 1$ throughout U, whence $U \subset C$ so that m cannot belong to ∂C. This proves our claim.

The hypothesis of Proposition 1 is precisely that $C \neq \phi$. The conclusion is that $C = M$, so let us suppose the contrary, $C \neq M$. Then by (9), $\partial C \neq \phi$. Clearly $m \in \partial C$ implies $\mu(m) \geq 1$ by the upper semi-continuity of μ. The rest of the proof breaks down into two cases:

(a) $\mu(m) = 2$ for all $m \in \partial C$;

(b) $\mu(m) = 1$ for some $m \in \partial C$.

Let us start with assuming case (a). Select some $m \in \partial C$ and a coordinate neighborhood N of m with complex analytic coordinate z. By the remark above, $C \cap N \neq \phi$ and $C_0 \cap N \neq \phi$. Whenever we speak of the complex structure on M we refer to that obtained from the Riemannian structure. Define the vector fields v_1 and v_2 on N by

$$\partial/\partial z = v_1 - iv_2 , \qquad \partial/\partial \bar{z} = v_1 + iv_2 .$$

Then $\langle v_1, v_2\rangle = 0$ and $\|v_1\| = \|v_2\|$. Let f_1, f_2 be an orthonormal basis for B, and

$$z_j = \langle v_1, f_j\rangle - i\langle v_2, f_j\rangle , \qquad j = 1, 2 . \tag{10}$$

These functions z_j, as well as the functions z_α appearing in the next proposition, are in fact components with respect to some system of complex coordinates in $G_{n,2}$ of the complex conjugate of the generalized Gauss map

$$G: M \to G_{n,2}$$

introduced in [1]. Here $G_{n,2}$ is the Grassmann manifold of oriented planes through the origin in $\boldsymbol{R}^n$, which is a complex manifold of dimension $n - 2$. It is proved in [1] that G is an anti-holomorphic mapping if the immersion of M is minimal. This will also follow from our calculations:

For $\partial z_j/\partial \bar{z} = (v_1 + iv_2)(z_j) = \langle \nabla_1 v_1 + \nabla_2 v_2, f_j\rangle = -\langle v_1, v_1\rangle\langle H, f_j\rangle$ by Lemma 1. Hence $\partial z_j/\partial \bar{z}$ is zero at all points where $H(m) = 0$. In particular, let $E = \{m \in M \mid H(m) \neq 0\}$ and $D = M - \bar{E}$. Both D and E are open. Then z_1 and z_2 are holomorphic on $N \cap D$. We shall need the following lemma:

Lemma 3.

(i) $(z_1, z_2) = (0, 0)$ *if and only if* $\mu(m) = 2$.

(ii) *If* z_1/z_2 *or* z_2/z_1 *is defined and real at* $m \in N$, *then* $\mu(m) = 1$, *and conversely.*

Proof. (i) We have $(z_1, z_2) = (0, 0)$ if and only if $\langle v_1, f_1\rangle = \langle v_2, f_1\rangle = \langle v_1, f_2\rangle = \langle v_2, f_2\rangle = 0$, which is true if and only if $v_1, v_2 \in A$, which is equivalent to $\mu(m) = 2$. (ii) Suppose that $z_2 \neq 0$ and that z_1/z_2 is real at m. It is easy to see that a rotation of the axes along v_1 and v_2 through an angle θ effects a multiplication of $\langle v_1, f_j\rangle - i\langle v_2, f_j\rangle$ by the number $e^{i\theta}$. Hence z_1/z_2 will be left unchanged by such a rotation. Let us rotate v_1 and v_2 if necessary to obtain a real value for z_2. But then z_1 must also be real, so that $\langle v_2, f_1\rangle = \langle v_2, f_2\rangle = 0$. This implies that v_2 is orthogonal to B and thus belongs to A, so that $\mu(m) = 1$, since $\mu(m) = 2$ is already excluded by (i). To get the converse to (ii) suppose $\mu(m) = 1$. Assume $z_2 \neq 0$. Applying a rotation of the basis v_1, v_2 if necessary we may assume $v_2 \in A$. But then $\langle v_2, f_j\rangle = 0$ for $j = 1, 2$, so that

$$z_1/z_2 = \langle v_1, f_1\rangle / \langle v_1, f_2\rangle ,$$

which is real. This completes the proof.

We now return to the proof of Proposition 1. It is clear from the definitions of C, D, C_0, and E, and Lemma 2 that $C_0 \subset D$ and $E \subset C$. Define a function $\varphi\colon N \to C^2$ by

$$\varphi(z) = \begin{cases} (z_1, z_2) & \text{if } z \in C_0 \cap N , \\ (0, 0) & \text{if } z \in \bar{C} \cap N , \end{cases}$$

Then $\varphi(z)$ is continuous on N since $(z_1, z_2) = (0, 0)$ on $\partial(C \cap N)$ by our assumption in case (a). Also $\varphi(z)$ is holomorphic where it is different from zero. Hence by a theorem of Rado [2] $\varphi(z)$ is holomorphic on N. Since $C \cap N$ is nonempty this implies $\varphi(z) = (0, 0)$. Hence $\mu(m) = 2$ throughout $C_0 \cap N$, contradicting the definition of C_0, and completing part (a) of the proof.

We now turn to case (b) where we assume that there exists a point $m \in \partial C$ such that $\mu(m) = 1$. We select a neighborhood N of m with coordinate z and define functions z_1 and z_2 all exactly as above. However, in this case we may assume that $\mu \leq 1$ and $z_2 \neq 0$ throughout N, because of the semi-continuity of μ and the continuity of z_2 respectively. Hence $\mu \equiv 1$ on $C \cap N$. Let $w(z) = z_1/z_2$ on N and define the function $\psi\colon N \to C$ by

$$\psi(z) = \begin{cases} \partial w/\partial z & \text{on } C_0 \cap N , \\ 0 & \text{on } \bar{C} \cap N . \end{cases}$$

We claim that ψ is continuous on N. In fact let $w = u + iv$. Then

$$\begin{aligned}\partial w/\partial z &= (1/2)((\partial/\partial x) - i(\partial/\partial y))(u + iv) \\ &= (1/2)((\partial u/\partial x) + (\partial v/\partial y)) + (i/2)((\partial v/\partial x) - (\partial u/\partial y)) \\ &= (\partial v/\partial y) + i(\partial v/\partial x) \qquad \text{on } C_0 \cap N \subset D\,.\end{aligned}$$

On the other hand,

$$(\partial v/\partial y) + i(\partial v/\partial x) = 0 \qquad \text{on } C \cap N\,,$$

since by Lemma 3 above $v \equiv 0$ there. Therefore $\partial w/\partial z \to 0$ as $z \to \partial(C \cap N)$ so that ψ is continuous. But ψ is holomorphic on $C_0 \cap N \subset D \cap N$, so that by Rado's theorem [2], $\psi \equiv 0$ on N. Therefore z_1/z_2 is constant on $C_0 \cap N$. But by Lemma 3, z_1/z_2 is real on $\partial(C_0 \cap N)$, so that z_1/z_2 must be real on $C_0 \cap N$. Hence, by Lemma 3, $\mu = 1$ throughout N, again a contradiction, and Proposition 1 is proved.

Proposition 2. *If* (8) *holds and* $\mu(m) = 1$ *on some open set, then* $\mu(m) = 1$ *throughout* M, *except at isolated points where* $\mu(m) = 2$.

Note that in the case of codimension one, $\mu = 0$ or 1 on M so that these isolated points cannot occur,

Proof. We already know by Proposition 1 that $\mu(m) \geq 1$ throughout M. Let $m_1 \in M$ be such that $\mu(m_1) = 2$. We only need to show that m_1 has a neighborhood U in which $\mu(m) = 1$ for all $m \neq m_1$. Let N be a coordinate neighborhood of m_1 with complex coordinate z, and f_α, $\alpha = 3, \cdots, n$, a fixed orthonormal basis for A. Define on N the functions z_α by

$$z_\alpha = \langle v_1, f_\alpha\rangle - i\langle v_2, f_\alpha\rangle\,, \qquad \alpha = 3, \cdots, n\,. \tag{11}$$

By a calculation similar to that used for (10), we get, for each α,

$$\partial z_\alpha/\partial \bar{z} = -\langle v_1, v_1\rangle\langle H, f_\alpha\rangle = 0\,,$$

since $H \perp f_\alpha$ by (8). In other words, the functions z_α are holomorphic throughout N. But then so is $\lambda(z) = z_3^2 + \cdots + z_n^2$. We claim that

$$\lambda(z) = \|\rho(v_1)\|^2 - \|\rho(v_2)\|^2 - 2i\langle\rho(v_1), \rho(v_2)\rangle\,, \tag{12}$$

where ρ is the orthogonal projection of $\boldsymbol{R}^n$ onto A. For

$$\begin{aligned}\lambda(z) &= \sum_\alpha \{\langle v_1, f_\alpha\rangle^2 - \langle v_2, f_\alpha\rangle^2\} - 2i\sum_\alpha \langle v_1, f_\alpha\rangle\langle v_2, f_\alpha\rangle \\ &= \|\rho(v_1)\|^2 - \|\rho(v_2)\|^2 - 2i\langle v_1, \rho(v_2)\rangle \\ &= \|\rho(v_1)\|^2 - \|\rho(v_2)\|^2 - 2i\langle\rho(v_1), \rho(v_2)\rangle\,.\end{aligned}$$

We now claim that

$$\lambda(z) = 0 \Longleftrightarrow \mu(m(z)) = 2 . \tag{13}$$

For let π be the orthogonal projection onto B as usual. That $\mu(m) = 2$ implies $\lambda(z) = 0$ is clear from (12). Conversely, if $\lambda(z) = 0$, we may write $v_1 = \pi v_1 + \rho v_1$ and $v_2 = \pi v_2 + \rho v_2$, and note that since $\mu(m) \geq 1$, $\dim \pi(T_m) \leq 1$ so that πv_1 and πv_2 are linearly dependent. But

$$0 = \langle v_1, v_2 \rangle = \langle \pi v_1, \pi v_2 \rangle + \langle \rho v_1, \rho v_2 \rangle ,$$

and we are given

$$0 = \langle \rho v_1, \rho v_2 \rangle ,$$

since $\lambda(z) = 0$, whence

$$0 = \langle \pi v_1, \pi v_2 \rangle ,$$

which implies that one of πv_1 and πv_2 vanishes. Say $\pi v_1 = 0$. That is, $v_1 \in A$. But then $\| v_1 \| = \| v_2 \|$ and $\| \rho v_1 \| = \| \rho v_2 \|$ together imply $v_2 \in A$ as well. This proves our claim (13), but also completes the proof of Proposition 2, for $\lambda(z)$ is holomorphic on N with zeros corresponding precisely to the points where $\mu(m) = 2$. Hence such points are isolated; in particular, m_1 is.

Proposition 3. *If* (8) *holds and* $\mu(m) = 2$ *on some open set, then* $\mu(m) = 2$ *throughout* M.

Proof. By Proposition 1 we then have $\mu(m) \geq 1$ everywhere. Suppose there is a point m where $\mu(m) = 1$. By the semi-continuity of μ this must remain true in a neighborhood of m. But then by Proposition 2, $\mu(m) = 2$ only at isolated points. This contradicts our hypothesis, and thus completes the proof.

3. The proof of the theorem

To begin the proof of the theorem, note that by the semi-continuity of μ the set C_0 of points where $\mu(m) = 0$ is open in M. If C_0 is not also dense in M, Proposition 1 shows that C_0 must be empty. In that case either $\mu \equiv 2$ or else by Proposition 2, $\mu(m) = 1$ except at isolated points m_i for which $\mu(m_i) = 2$. Thus there are only three possibilities:

(i) $\mu(m) = 0$ on an open dense subset C_0 of M;

(ii) $\mu \equiv 2$ on M;

(iii) $\mu(m) = 1$ on $M_0 = M - \{m_i\}$, where $\{m_i\}$ is a set of isolated points at which $\mu(m) = 2$.

If (i) holds, we see by Lemma 2 that $H(m) = 0$ on C_0 and hence, by the continuity of H, $H \equiv 0$ on M so that $x \colon M^2 \to \boldsymbol{R}^n$ is a minimal immersion, as in case 1 of the theorem.

If (ii) holds, the immersion $x \colon M^2 \to \boldsymbol{R}^n$ is always tangent to a translate A' of A, so that we may regard it as an immersion into A'. But then the hypothesis (2) says that it is a minimal immersion in A'. Since A' is totally geodesic in $\boldsymbol{R}^n$, then $x\colon M^2 \to \boldsymbol{R}^n$ must itself be a minimal immersion so that we are again in case 1 of the theorem.

If (iii) holds, we shall show that we get case 2. In this case each point $m \in M_0$ has a neighborhood U on which we can define a C^∞ vector field X_A such that $X_A \in T_m \cap A$ and $\|X_A\| = 1$. X_A is of course unique up to sign. Using the trajectories of X_A as lines $\xi =$ a constant and the orthogonal trajectories as lines $\eta =$ a constant we obtain a local coordinate system (ξ, η) so that $\langle \partial/\partial\xi, \partial/\partial\eta \rangle = 0$, $\partial/\partial\eta \in A$ and $\partial/\partial\xi \notin A$. But then any curve $\xi =$ constant is everywhere tangent to A so that π maps such a curve into a single point in B. On the other hand, the curves $(\cdot, \eta_1)$ and $(\cdot, \eta_2)$ are mapped into the same curve $\gamma(\xi) = \pi(\xi, \eta_1) = \pi(\xi, \eta_2)$ on B, and $\gamma(\xi)$ has a nonvanishing tangent everywhere since $\partial/\partial\xi \notin A$. Hence, by changing the parameter ξ if necessary we may assume that $\|(\partial/\partial\xi)(\gamma(\xi))\| = 1$ always, that is, $\|\pi_*(\partial/\partial\xi)\| = 1$. Thus the neighborhood U projects onto the curve $\gamma(\xi)$ in B. Hence $U \subset \gamma \times A$. In fact, $U \subset \gamma \times A$ as a minimal surface, for we already know that $H \perp A$, and clearly $H \perp (\partial/\partial\xi)$. Together they imply $H \perp \gamma \times A$ so that U is a minimal surface in $\gamma \times A$. As it is easy to show that immersions of type 1 or 2 satisfy (2), the proof of the theorem is now complete.

Proof of the corollary. Embed $\boldsymbol{R}^n$ as a linear subspace of $\boldsymbol{R}^{n+1}$. Then $x\colon M^2 \to \boldsymbol{R}^{n+1}$ satisfies the hypothesis of the theorem with $n+1$ in place of n. Let B be the orthogonal complement of A in $\boldsymbol{R}^{n+1}$, and $\pi\colon \boldsymbol{R}^{n+1} \to B$ the orthogonal projection. If x is of type 1 and thus minimal into $\boldsymbol{R}^{n+1}$, it is also minimal into $\boldsymbol{R}^n$ since it already maps into the latter space. It remains to show that under the hypothesis an immersion of type 2 is also minimal. Suppose $x\colon M^2 \to \boldsymbol{R}^{n+1}$ is of type 2. Then except for isolated points $\{m_i\}$, each $m \in M$ has a neighborhood U for which $\pi(x(U))$ is a regular curve γ in B. Let B_1 be the orthogonal complement of $\boldsymbol{R}^n$ in $\boldsymbol{R}^{n+1}$, and B_2 the orthogonal complement of A in $\boldsymbol{R}^n$. Then $B = B_1 \times B_2$. But $x(U) \subset \boldsymbol{R}^n$ implies that $\gamma \subset B_2$. Since B_2 is one-dimensional, and γ regular, γ must be an open interval I in B_2. Hence $x \mid U$ is minimal into $I \times A$, and thus into $\boldsymbol{R}^n$. Hence the mean curvature vector H vanishes on U. It follows by the continuity of H that $x\colon M^2 \to \boldsymbol{R}^n$ is minimal.

4. Example

In the introduction we promised an example of an immersion $x\colon M^2 \to \boldsymbol{R}^4$ of type 2 for which the description given in the theorem is the best possible. In particular, there will be two points m_1 and m_2 on M which have no neighborhood U which π projects onto a regular curve in B, and hence no neighborhood on which x is minimal immersion into a piece of cylinder $\gamma \times A$.

To get such an example rotate the curve

$$x^1 = (1/2)(\exp(x^2) + \exp(-x^2))$$

about the x^2 — axis to get the catenoid $M^2 \subset \boldsymbol{R}^3$. Put

$$W = \boldsymbol{R} = \{x^1 = x^2 = 0\}$$

and identify $\boldsymbol{R}^3$ with $W \times \boldsymbol{R}^2$. Also let A and B be orthogonal 2-spaces in $\boldsymbol{R}^4$, so that $\boldsymbol{R}^4 = A \times B$. Let $\gamma^1(\xi)$ and $\gamma^2(\xi)$ be two C^∞ curves in B, $-\infty < \xi < \infty$, parametrized by the arc length, such that

$$\gamma^1(\xi) = \gamma^2(\xi) \qquad \text{for } -1 \leq \xi \leq 1 \, .$$

$\gamma^1 \circ x^3$ and $\gamma^2 \circ x^3$ are then immersions of W into B. Let i be any isometric isomorphism from $\boldsymbol{R}^2$ to A. Then $(\gamma^1 \circ x^3) \times i$ and $(\gamma^2 \circ x^3) \times i$ are both immersions of $\boldsymbol{R}^3 = W \times \boldsymbol{R}^2$ into $B \times A = \boldsymbol{R}^4$, which when restricted to $M^2 \subset W \times \boldsymbol{R}^2$ would give immersions of type 2 in $\boldsymbol{R}^4$. To get our example we assume in addition that $\gamma^1(\xi)$ and $\gamma^2(\xi)$ do not agree when $|\xi| > 1$, and then define an immersion $x\colon M^2 \to \boldsymbol{R}^4$ as follows:

$$x(m) = \begin{cases} (\gamma^1(x^3(m)), i(m)) & \text{on the half-space } x^2 \geq 0 \, , \\ (\gamma^2(x^3(m)), i(m)) & \text{on the half-space } x^2 \leq 0 \, . \end{cases} \tag{14}$$

To show that this defines a smooth immersion, let $m_1 = (0, 0, -1)$ and $m_2 = (0, 0, 1) \in M$, and let $m \in M_0 = M - \{m_1, m_2\}$. Then either $x^2(m) \neq 0$ so that only one half of the definition (14) applies in a neighborhood of m, whence x is smooth at m, or else $|x^3(m)| < 1$ so that the two halves of (14) agree completely on a neighborhood of m, so that x is smooth here also. That x is smooth at m_1 and m_2 as well is not difficult to see.

By its definition, $x\colon M^2 \to \boldsymbol{R}^4$ clearly belongs to type 2 in the theorem. In particular, it satisfies the hypothesis (2). Any neighborhood U of m_1 is mapped by π, the projection with kernel A, onto a subset of B containing the set

$$\{\gamma^1(\xi) \,|\, -1 - \varepsilon < \xi < -1 + \varepsilon\} \cup \{\gamma^2(\xi) \,|\, -1 - \varepsilon < \xi < -1 + \varepsilon\} \, ,$$

which is not a regular curve, as γ^1 and γ^2 were chosen so as to differ outside $\{|\xi| \leq 1\}$. The same holds true for the point m_2. Thus, for no neighborhood U of m_i, $i = 1, 2$, does $x|U$ map into a piece of cylinder $\gamma \times A$.

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